Phenomenology at colliders (2)

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Plan

I. INTRODUCTION AND MOTIVATION

II. STRUCTURE FUNCTIONS AND PARTON DISTRIBUTION FUNCTIONS

1. Deep inelastic scattering and structure functions
2. Quark parton model
3. Scaling violation
4. QCD evolution and DGLAP equations

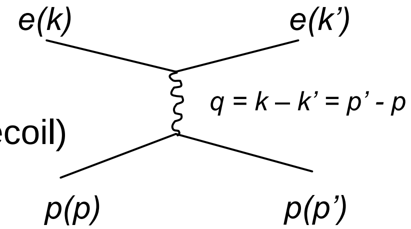
IiI. FACTORISATION THEOREMS; PDF PARAMETERISATIONS

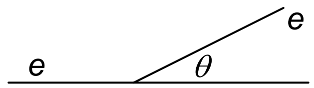
1. Factorisation theorems
2. Drell-Yan production with CMS
3. Parton distribution function parameterisations
4. Parton distribution uncertainties
5. Some (of many) uncovered topics

II. Structure functions and parton distribution functions

II.1 Deep inelastic scattering and structure functions

1. Rutherford formula





scattering of spin 0 (α particle) by spin 0 nucleus (no recoil)

$$\frac{d\sigma}{d\Omega} = Z^2 \frac{\alpha^2}{4E^2 \sin^4(\theta/2)} \bullet F(q^2) = \int \rho(R) e^{iqR} d^3R$$

form factor : extended target (Fourier transform of charge distribution)

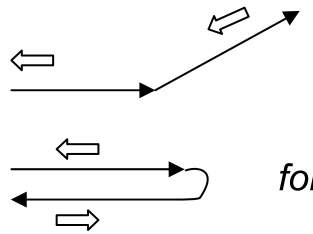
2. Mott formula

scattering of spin 1/2 (electron) by spin 0 nucleus (neglecting electron mass)

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4(\theta/2)} \frac{E'}{E} \cos^2(\theta/2)$$

Rutherford — electron spin

recoil mass



forbidden

3. Spin 1/2 – spin 1/2 point-like scattering (« e – μ »)

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4(\theta/2)} \frac{E'}{E} \left[\cos^2(\theta/2) + \frac{Q^2}{2M^2} \sin^2(\theta/2) \right]$$

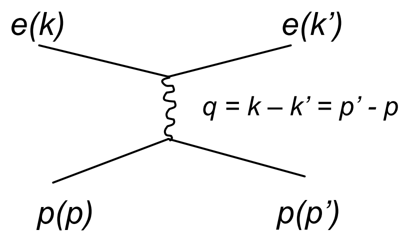
Mott — magnetic interaction ($\sigma^{\mu\nu}$)

$$= \frac{4\pi\alpha^2}{Q^4} (1 - y + y^2/2)$$ with $y = 1 - E'/E \cos^2(\theta/2)$ and $Q^2 = -q^2 = 4EE' \sin^2(\theta/2) \simeq p_T^2$

NB $q^2 < 0$: *scattering*

large p_T <-> small transverse distance

4. Extended spin 1/2 target ; form factors



$$M = \frac{4\pi\alpha^2}{Q^4} J_\mu^{(e)}(q)\, J^{\mu(p)}(q)$$

with $J_\mu^{(e)} = \bar{u}(k')\,\gamma_\mu\, u(k)$

$$J^{\mu(p)} = \bar{u}(p')\left[F_1(q^2)\,\gamma^\mu + i\frac{\kappa}{2M}F_2(q^2)\,q_\nu\,\sigma^{\mu\nu}\right]u(p)$$

covariance : $\gamma^\mu \quad q^\mu \quad q_\nu\,\sigma^{\mu\nu} \quad \oplus$ current conservation : $\partial_\mu J^{\mu(p)}(x) = 0 \Rightarrow q_\mu J^{\mu(p)}(q) = 0$

$F_1(q^2) \quad F_2(q^2)$: 1 invariant variable (+ trivial azimuthal angle) + CM energy $\sqrt{s}$

In practice, combine F_1 and F_2 → **G_E and G_M « form factors »**

→*Rosenbluth formula* for *e p* elastic scattering

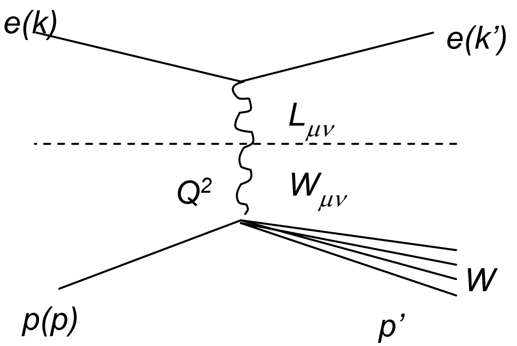
$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega}(Mott)\cdot\left[\frac{G_E^2 + (Q^2/4M^2)\,G_M^2}{1+Q^2/4M^2} + \frac{Q^2}{4M^2}\,2\,G_M^2\,tg^2(\theta/2)\right]$$

Experimentally : $G_E \simeq G_M \simeq \dfrac{1}{(Q^2+0.71^2)^2}$ (dipole parameterisation)

i.e. ***$1/Q^8$*** compared to point-like target !

photon wave length decreases with Q^2 => probability to break proton increases
(the photon wave must embrace coherently the whole proton)

5. Deep inelastic scattering



$$M \sim \left[J_\mu^{(e)}(q)\, J^{\mu(p)}(q) \right] + cc \quad = L_{\mu\nu}\, W^{\mu\nu}$$

$$L_{\mu\nu} = 2k_\mu k'_\nu + 2k'_\mu k_\nu - Q^2 g_{\mu\nu} \qquad \text{(for em interactions, } \gamma \text{ exchange)}$$

$$W^{\mu\nu} = -W_1\, g^{\mu\nu} + W_2\, p^\mu p^\nu + W_4\, q^\mu q^\nu + W_5\, (p^\mu q^\nu + q^\mu p^\nu)$$

current conservation $q_\nu W^{\mu\nu} = q_\mu W^{\mu\nu} = 0 \quad \Rightarrow$ only 2 of the 4 W functions contribute : W_1 and W_2

proton dissociation → one additional invariant *W* in addition to Q^2 (and *s*)

→ $W_{1,2}(Q^2, W)$ or any combination, in particular $x = Q^2 / 2\, p.q = Q^2 / 2M\nu$ in proton rest frame

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4(\theta/2)} \frac{E'}{E} \left[W_2(\nu, Q^2) \cos^2(\theta/2) + 2W_1(\nu, Q^2) \sin^2(\theta/2) \right]$$

$W_{1,2}(Q^2, W)$: <u>*physical observables*</u> (measured quantities)

DIS cross section

$$F_1(x,Q^2) = MW_1 \qquad F_2(x,Q^2) = \nu W_2$$

$$\frac{d^2\sigma}{dxdy} = \frac{d^2\sigma}{dxdQ^2}xs = \frac{4\pi\alpha^2}{Q^4}s\left[(1-y)F_2(x,Q^2) + \frac{y^2}{2}2xF_1(x,Q^2)\right] \qquad \text{em interaction : NC } \gamma \text{ exchange}$$

$$= \ldots\left[\ldots \pm \frac{G_F^2}{8\pi^2}\frac{Q^4}{(1+Q^2/M^2)}y(1-y/2)\,xF_3(x,Q^2)\right] \qquad \text{weak interaction : CC } W \text{ exchange}$$

F_1, F_2, $F_3(x,Q^2)$ = structure functions – physical observables (measured quantities)

In the following, we shall concentrate on the structure function behaviour,
but don't forget the $1/Q^4$ factor in the cross section !

For completeness (bis)

"polarised cross sections" (with different y dependences)

$$\sigma_T = \frac{1}{2}(\sigma^+ + \sigma^-) = \frac{4\pi\alpha^2}{s}\, 2\, F_1 \qquad \sigma_L = \sigma^0 = \frac{4\pi\alpha^2}{sx}(F_2 - 2\, xF_1) \qquad F_L = \frac{1}{2x}(F_2 - 2xF_1)$$

σ_T and σ_L can be separated by different dependences in y

Kinematical variables definitions and relations

$$s = (p+k)^2 \qquad Q^2 = -q^2 \qquad \nu = p.q\,/\,2M \qquad x = Q^2\,/\,2M\,\nu \qquad y = \frac{p.q}{p.k}$$

$$W^2 \;=\; Q^2\,(1/\,x\,\text{-}\,1) \;+\; M^2 \simeq Q^2\,/\,x \simeq y\;s \qquad Q^2 \simeq x\;y\;s \qquad 0 \le x \le 1 \quad 0 \le y \le 1$$

Laboratory frame $\quad \nu = E - E' \qquad y = \dfrac{\nu}{E} \qquad Q^2 = 4EE'\sin^2(\theta/2) \qquad y \;=\; 1 - E'/E\cos^2(\theta/2)$

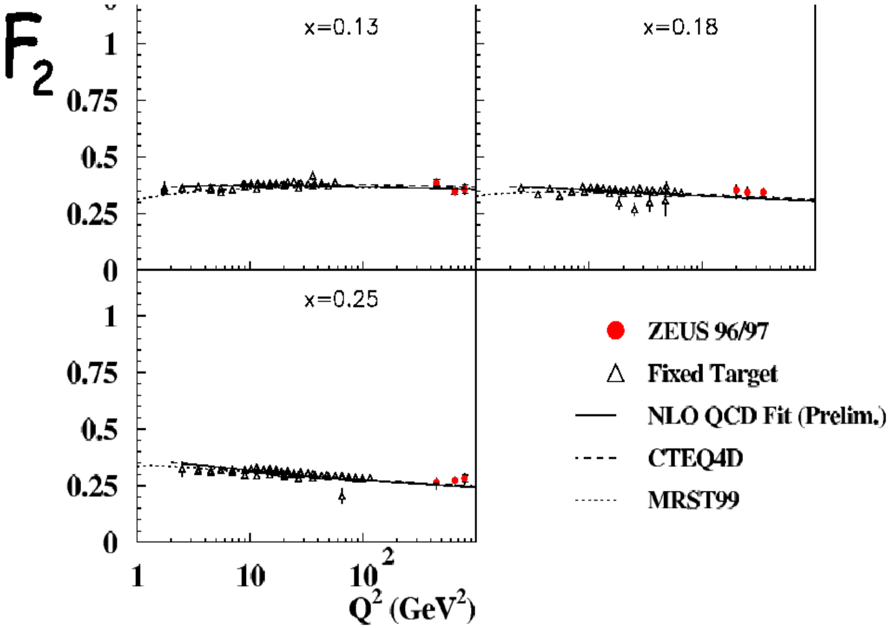
CM frame $\quad 1 - y = \dfrac{1}{2}(1+\cos\theta^*) \quad \rightarrow$ controls helicity : $y = 1 \leftrightarrow$ backward scattering, forbidden for long. photon

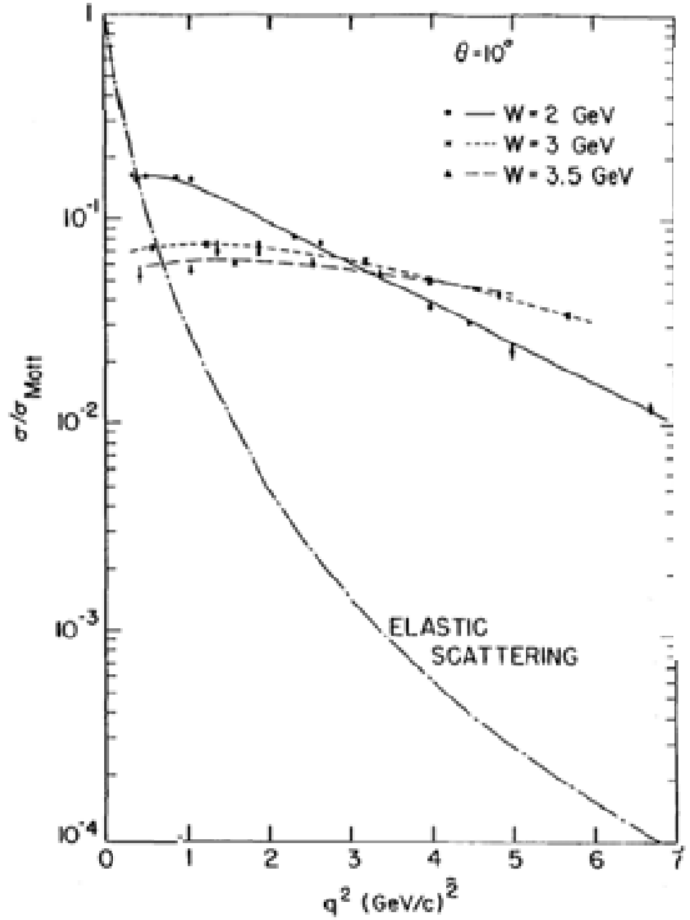
NB At high Q^2 (HERA), also NC Z exchange + γ–Z interference

II.2 Quark parton model

Scaling

ep scattering SLAC 1969, for sufficient energy and Q^2 :
observation of approximate « **scaling** »
i.e. no strong Q^2 dependence of cross section
(except for common $1/Q^4$)





Point-like partons

Compare

$$\frac{d^2\sigma}{d\Omega dE'} = \frac{4\alpha^2 E'^2}{Q^4}\left[\cos^2(\theta/2) + \frac{Q^2}{2M^2}\sin^2(\theta/2)\right]\delta(\nu - \frac{Q^2}{2M}) \quad (1)$$ elastic scattering on point-like spin 1/2 target

$$= \frac{4\alpha^2 E'^2}{Q^4}\left[W_2(\nu,Q^2)\cos^2(\theta/2) + 2W_1(\nu,Q^2)\sin^2(\theta/2)\right] \quad (2)$$ deep inelastic scattering

If DIS is in fact **elastic scattering** on **spin 1/2 pointlike** « partons » with charge *e*, momentum *p*, mass *m*,

then one has (using $\delta(x/a) = a\,\delta(x)$)

$$\nu W_2(\nu,Q^2) = e^2\,\delta(1-\frac{Q^2}{2m\nu}) \qquad 2mW_1(\nu,Q^2) = e^2\frac{Q^2}{2m\nu}\,\delta(1-\frac{Q^2}{2m\nu}) \qquad (3)$$

→ scattering on _partons_ explains _scaling_,

i.e. the fact that the structure functions $W_{1,2}$ depend on one variable only :

$\boldsymbol{x = Q^2/2m\nu}$ – or equivalently W or ν –

and not separately on Q^2 and ν

Interpretation of the x variable

if masses and transerse momenta can be neglected

-> in the proton rest frame

quark : (ξM 0 0 0)

photon : (ν 0 0 $(Q^2+\nu^2)^{1/2}$) since $q^2 = -Q^2 = \nu^2 - p_g^2$

=> $2\,\xi\,M\,\nu - Q^2 = m_q^2 = 0$

x = ξ = fraction of proton momentum carried by the quark

***x* is the momentum fraction of the proton carried by the struck quark**

Rigourously, in the Breit frame, i.e. where photon is purely space-like :

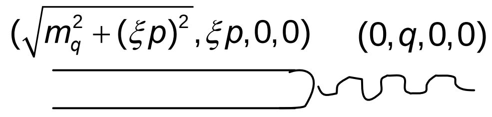
$$W^2 = 2M\nu \simeq Q^2 / x$$

Interpretation of the x variable (more rigourously)

Let the hit quark carry be parallel to the proton and carry

the fraction ξ of the proton momentum p

In the Breit frame, i.e. where photon is purely space-like :



initial state inv. mass $= m_q^2 + (\xi p)^2 - (\xi p + q)^2$

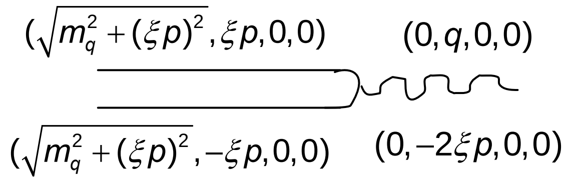
$$= m_q^2 + (\xi p)^2 - (\xi p)^2 - 2\xi p.q - q^2 = m_q^2 \text{ (final state quark)}$$

$$\Rightarrow \xi = \frac{Q^2}{2p.q} = x$$

***x* is, in the Breit frame, the momentum fraction of the proton carried by the struck quark**

NB Breit frame is also called the « brick wall » frame :

$$Q^2 = 2\xi p.q \quad \Rightarrow \quad q = -2\,\xi p$$



More generally : x = fraction of proton momentum carried by the quark in *IFM* (**infinite momentum frame**), where masses and transerse momenta can be neglected

NB $\quad W^2 \simeq Q^2 / x$

Parton distribution functions

***Incoherent* scattering** on constituent partons, « **frozen** » in the proton by **time dilatation** (NB also longitudinal contraction) :

parton-parton interaction time $\sim \gamma / R_p$ $\ll$ high energy γ p interaction time

hence : $$\sigma(e\,p) = \sum_i \int dx\, f_i(x)\, \sigma(e\,q_i) \qquad (4)$$

where ***$f_i(x)$* = probability to find in the proton parton of species *i* carrying momentum fraction *x* (in IMF)**

NB : f_i = valence + sea

$$\nu W_2(\nu, Q^2) = e^2\, \delta(1 - \frac{Q^2}{2m\nu})$$

$$MW_1(\nu, Q^2) = e^2 \frac{Q^2}{2m\nu}\, \delta(1 - \frac{Q^2}{2m\nu})$$

$$\left.\begin{array}{l} \rightarrow F_2(x) = \sum_i e_i^2\, x\, f_i(x) \\ \rightarrow F_1(x) = \frac{1}{2x} F_2(x) \end{array}\right\} \text{QPM}$$

Structure functions depend only on x

cross section given by quark distributions f(x)

$$\frac{d^2\sigma}{dxdy} = \frac{2\pi\alpha^2}{Q^4} s \left[1 + (1-y)^2\right] \sum_i e_i^2\, x\, f_i(x) \qquad \text{QPM}$$

Note that in QPM $F_2 = 2xF_1$ (Callan Gross relation)

Parton distribution functions (bis)

Incoherent scattering on constituent partons, « frozen » in the proton by time dilatation (NB also longitudinal contraction) :

parton-parton interaction time $\sim \gamma / R_p \quad \ll \quad$ high energy γ p interaction time

hence : $\sigma(e\,p) = \sum_i \int dx\, f_i(x)\, \sigma(e\,q_i) \quad (4)$

where $f_i(x)$ = probability to find in the proton parton of species i carrying momentum fraction x (in IMF)

NB : f_i = valence + sea

Using $p_i = x\,P_p$ and thus formally $m = xM$ (= 0 in IMF !), putting in (4) the $W_{1,2}$ structure functions (3) and integrating over the δ function, only an x dependence remains at high energy, high Q^2 (*DIS regime*)

$$\left.\begin{aligned} \nu W_2(\nu,Q^2) &= e^2\,\delta\!\left(1-\frac{Q^2}{2m\nu}\right) &&\rightarrow F_2(x) = \sum_i e_i^2\, x\, f_i(x) \\ MW_1(\nu,Q^2) &= e^2\,\frac{Q^2}{2m\nu}\,\delta\!\left(1-\frac{Q^2}{2m\nu}\right) &&\rightarrow F_1(x) = \frac{1}{2x} F_2(x) \end{aligned}\right\} \text{QPM}$$

$$\frac{d^2\sigma}{dxdy} = \frac{2\pi\alpha^2}{Q^4}\, s\left[1+(1-y)^2\right] \sum_i e_i^2\, x\, f_i(x) \qquad \text{QPM}$$

Note that in QPM $F_2 = 2xF_1$ (Callan Gross relation)

→ measurement of $F_L = 0$ indicates that partons are massless spin 1/2 objects → identified with quarks

(Note also if quark spin were 0, $\sigma_T = 0$ – cf. Mott formula)

Sum rules, first pdf measurements

$$\sum_i \int dx\, x\, f_i(x) = 1$$ momentum conservation

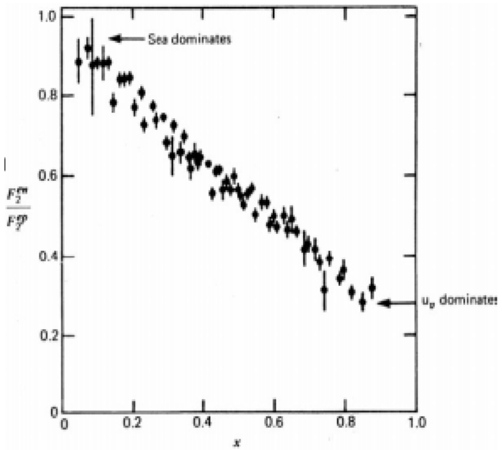
$$\int_0^1 dx\,[u(x)-\bar{u}(x)] = 2 \quad \int_0^1 dx\,\left[d(x)-\bar{d}(x)\right] = 1$$ valence quarks $$\int_0^1 dx\,[s(x)-\bar{s}(x)] = 0$$ sea - idem for c, b, t

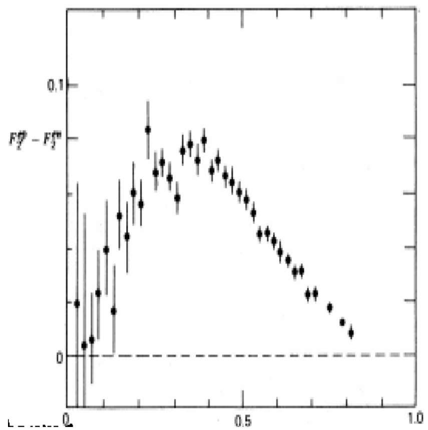
$$\frac{1}{x}F_2(x) = \sum_q e_q^2\, x\, f_q(x)$$ proton $\leftrightarrow$ neutron : $u(x) \leftrightarrow d(x)$ (isospin)

$$\frac{1}{x}F_2^{ep}(x) = \frac{4}{9}u_V + \frac{1}{9}d_V + \left(\sum_{SEA} e_q^2\, S(x)\right) \qquad \frac{1}{x}F_2^{en}(x) = \frac{1}{9}u_V + \frac{4}{9}d_V + \left(\sum_{SEA} e_q^2\, S(x)\right)$$

$$\frac{F_2^{en}(x)}{F_2^{ep}(x)} \xrightarrow[x\to 0]{} 1 \qquad \frac{F_2^{en}(x)}{F_2^{ep}(x)} \xrightarrow[x\to 1]{} \frac{u_V + 4d_V}{4u_V + d_V}$$

$$\frac{1}{x}\left(F_2^{ep}(x) - F_2^{en}(x)\right) = \frac{1}{3}\left(u_V(x) - d_V(x)\right)$$





- Fixed target electron and muon scattering on hydrogen and nuclei → F_2 for p and n

- neutrino scattering → F_2 and xF_3

$$F_2^{\nu+\bar{\nu}} = x\sum_q (q(x)+\bar{q}(x)) \qquad F_2^{\nu-\bar{\nu}} = x\sum_d d^V(x) - x\sum_u u^V(x)$$

$$F_3^{\nu+\bar{\nu}} = \sum_q (q(x)-\bar{q}(x)) = \sum_q q^V(x) \qquad F_3^{\nu-\bar{\nu}} = \sum_d (d(x)+\bar{d}(x)) - \sum_u (u(x)+\bar{u}(x))$$

- charm production by neutrinos → strange sea

- Using $\int_0^1 dx\, F_2^{ep}(x)$ and $\int_0^1 dx\, F_2^{en}(x)$ $\rightarrow$ gluons $\simeq 0.46$ proton momentum

→ **first determinations of pdf's**

NB Remember that structure functions are observables, but **pdf's are « theoretical » quantities**

Measured quantities are cross sections / structure functions

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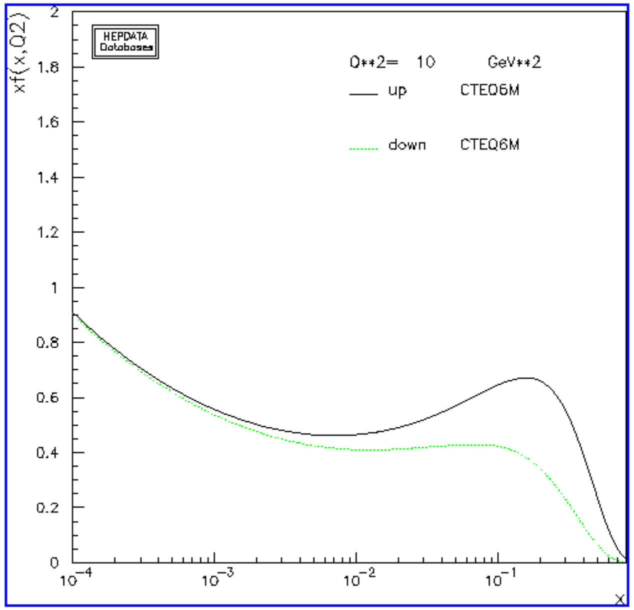

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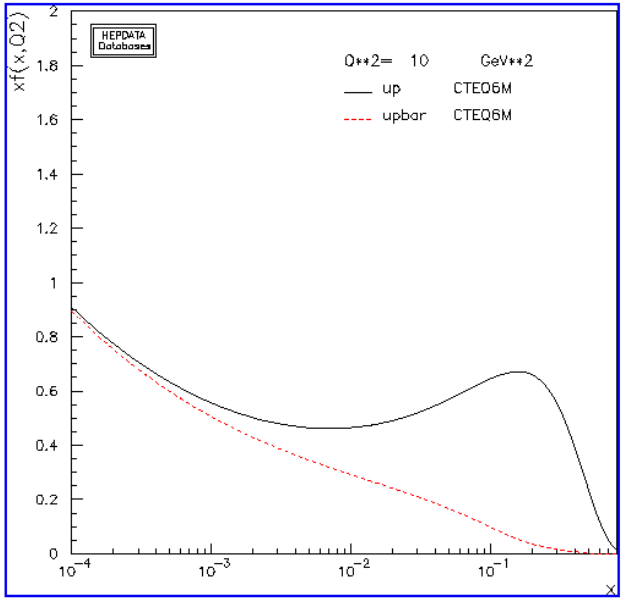

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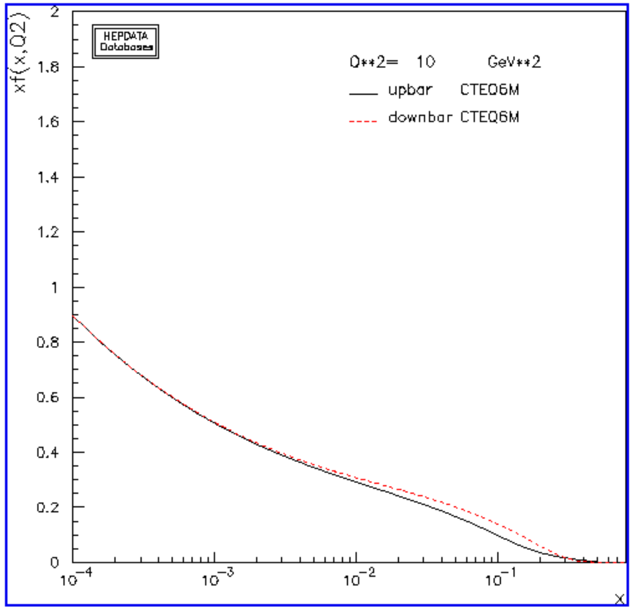
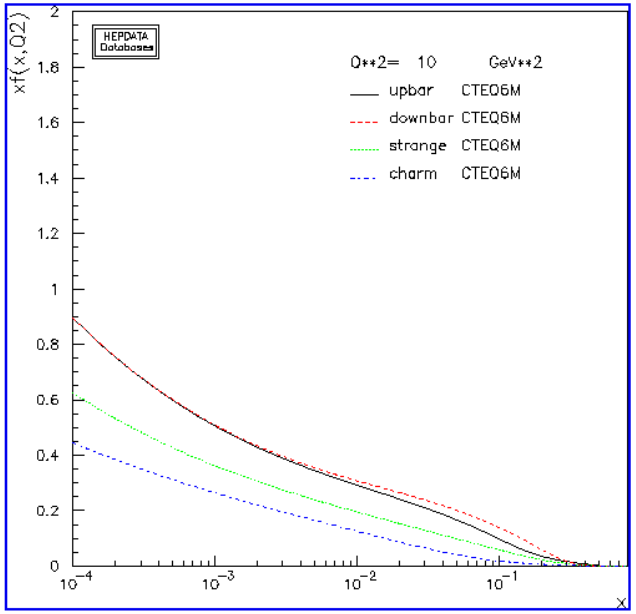
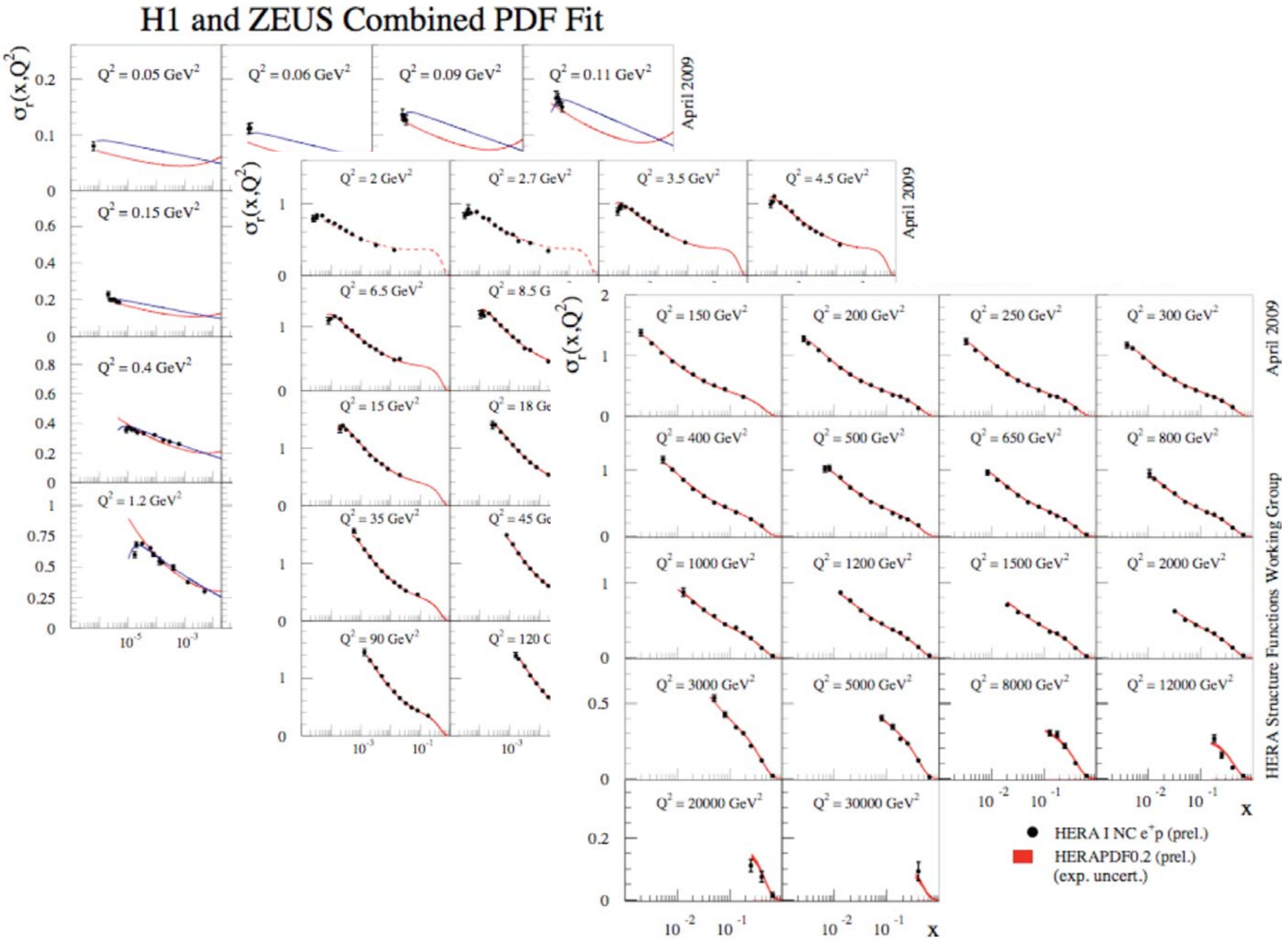


sea



http://durpdg.dur.ac.uk/hepdata/

II.3 Scaling violation



H1 and ZEUS Combined PDF Fit
April 2009
HERA Structure Functions Working Group
$\sigma_r(x,Q^2)$
$Q^2 = 0.05$ GeV^2
$Q^2 = 0.06$ GeV^2
$Q^2 = 0.09$ GeV^2
$Q^2 = 0.11$ GeV^2
$Q^2 = 0.15$ GeV^2
$Q^2 = 0.4$ GeV^2
$Q^2 = 1.2$ GeV^2
$Q^2 = 2$ GeV^2
$Q^2 = 2.7$ GeV^2
$Q^2 = 3.5$ GeV^2
$Q^2 = 4.5$ GeV^2
$Q^2 = 6.5$ GeV^2
$Q^2 = 15$ GeV^2
$Q^2 = 35$ GeV^2
$Q^2 = 90$ GeV^2
$Q^2 = 150$ GeV^2
$Q^2 = 200$ GeV^2
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$Q^2 = 1200$ GeV^2
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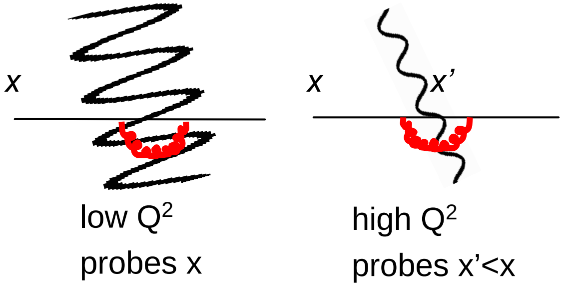
$Q^2 = 3000$ GeV^2
$Q^2 = 5000$ GeV^2
$Q^2 = 8000$ GeV^2
$Q^2 = 12000$ GeV^2
$Q^2 = 20000$ GeV^2
$Q^2 = 30000$ GeV^2
x
HERA I NC e^+p (prel.)
HERAPDF0.2 (prel.)
(exp. uncert.)

Scaling violations

Q^2 evolution of structure functions

photon resolution improves with Q^2

→ disentangles virtual gluon emission



As Q^2 increases,

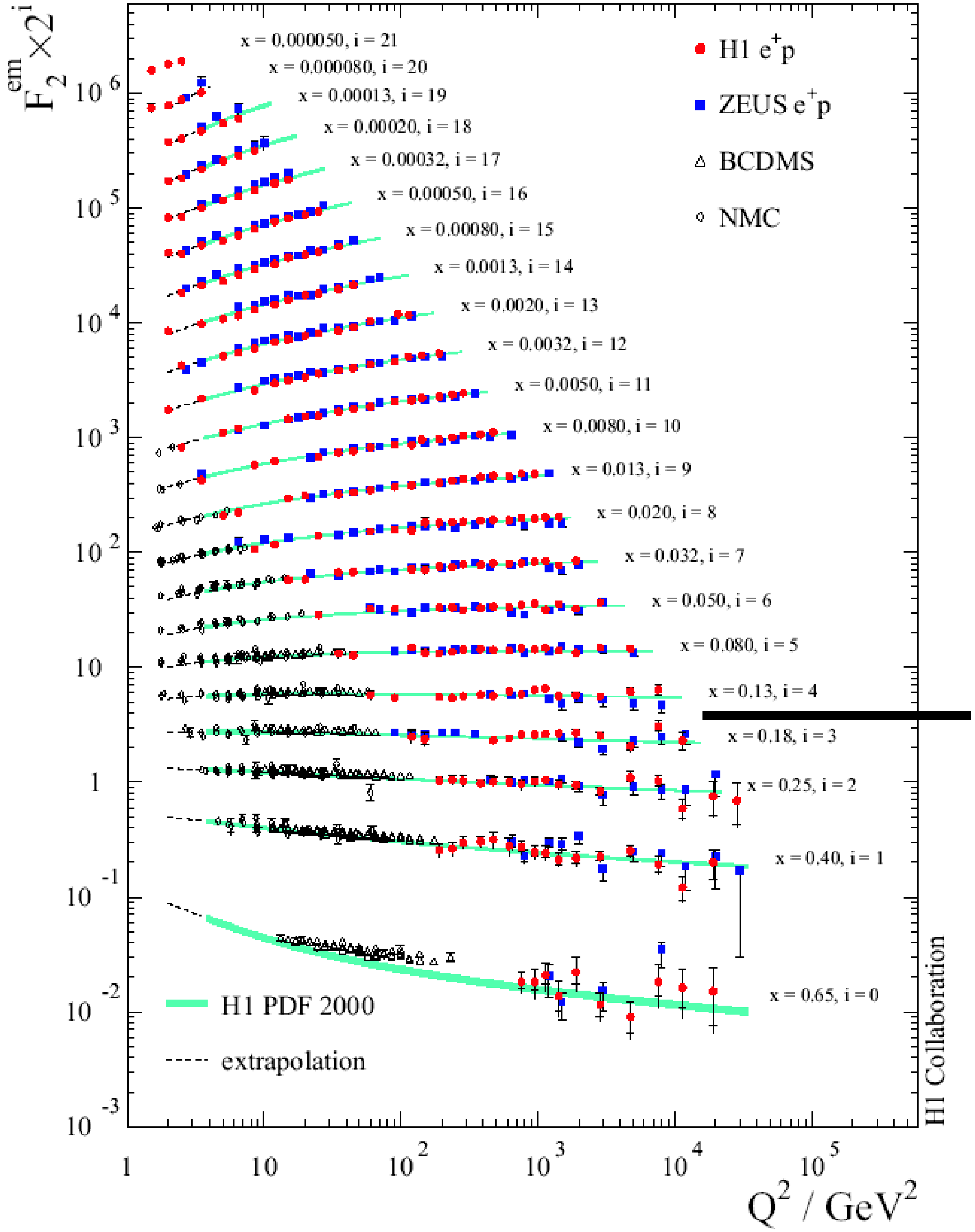
quark content decreases at large x (valence)

and increases at low x

also : at low x, the gluon content and the sea increase

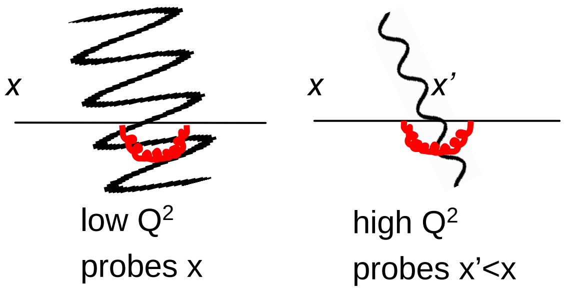
(low x since due to bremsstrahlung → soft)

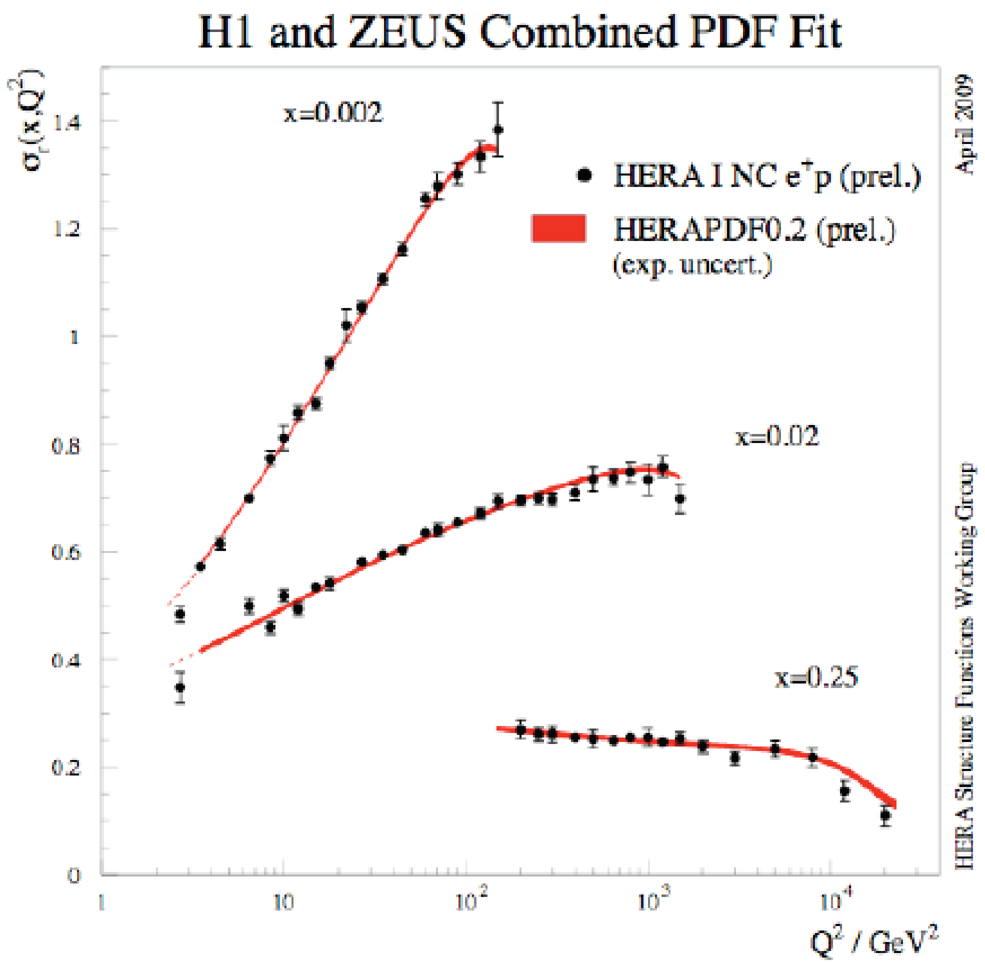
parton distribution function evolutions

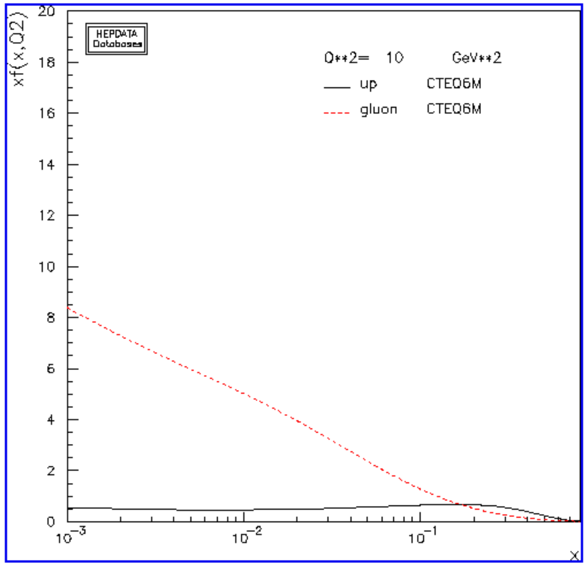


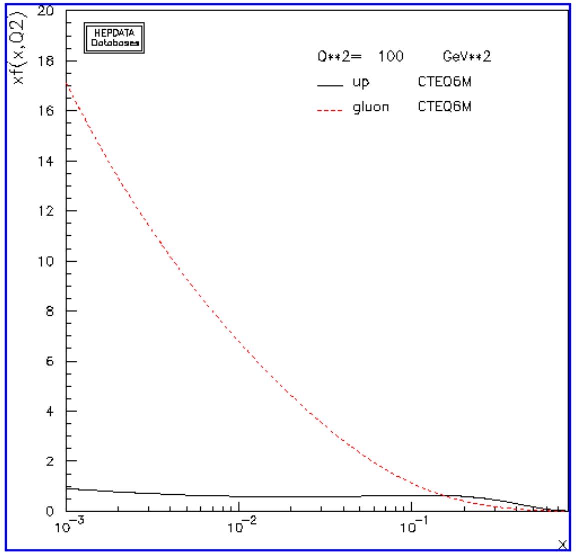
Q^2 evolution of the cross section (structure functions)

quark evolution

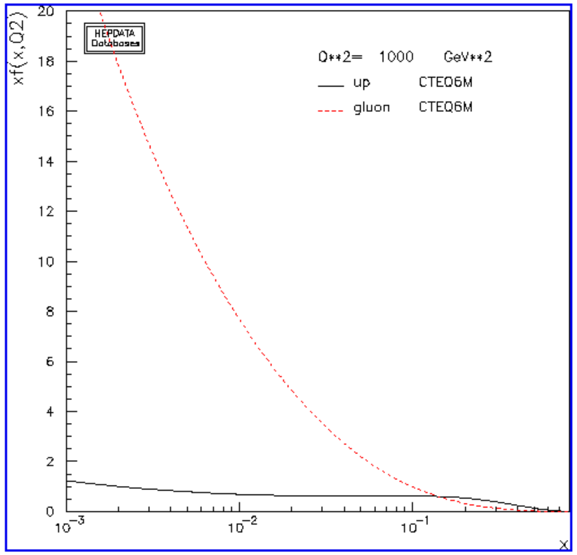








gluon evolution

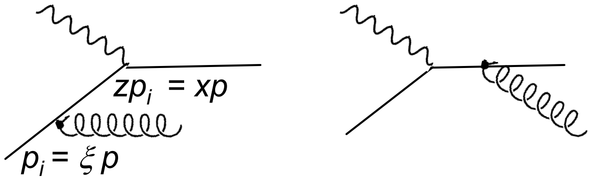


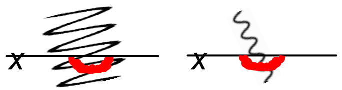
II.4 QCD evolution
DGLAP equations

« structure of the quark »

Gluon emission by the quark :

a quark « structure » shows up





low Q^2 probes x

high Q^2, probes x'< x

Take over the SF formalism, with proton → quark

	proton	quark
	p	$p_i = \xi p$
	$x = Q^2/2p.q$	$z = Q^2/2p_i.q = x/\xi$

Hence $$2F_1(x,Q^2) = \left.\frac{\sigma_T(x,Q^2)}{\sigma_0}\right|_{\gamma^* p} = \sum_i \int_0^1 dz \int_0^1 d\xi \, f_i(\xi)\, \delta(x-\xi z)\, \left.\frac{\hat{\sigma}_T(z,Q^2)}{\hat{\sigma}_0}\right|_{\gamma^* quark}$$

where $\sigma_0 = \frac{4\pi\,\alpha_S^2(Q^2)}{s}$ and similarly for $\hat{\sigma}_0$ with $\hat{s} = \xi s$

$f_i(\xi)$ is the probability to find in the proton a (« **primary** ») quark with momentum fraction ξ,

$\hat{\sigma}_T(z,Q^2)$ is the photon-quark transverse cross section,

for a (« **secondary** ») quark of momentum fraction ***z***;

ξ and z can vary from *0* to *1*, but **$x = \xi z$ is fixed** (hence the δ function)

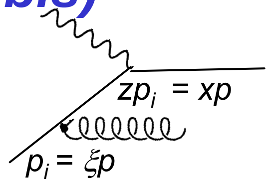
After integration on z :

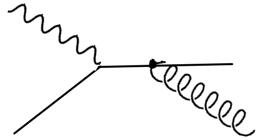
$$2F_1(x,Q^2) = \sum_i \int_0^1 \frac{d\xi}{\xi}\, f_i(\xi)\, \frac{\hat{\sigma}_T(x/\xi,Q^2)}{\hat{\sigma}_0}$$

« structure of the quark » (bis)

Gluon emission by the quark :

a quark « structure » shows up





NB 1. we consider *hard* gluon emission, over timescale comparable to interaction time → large p_T, well separated jets ←→ soft gluon emission during hadronisation (see later)

2. « before » and « after » are frame dependent - the second diagram for gauge invariance

Take over the SF formalism, with

proton	→	quark
p		$p_i = \xi p$
$x = Q^2/2p.q$		$z = Q^2/2p_i.q = x/\xi$

Hence $$\frac{1}{x}F_2(x,Q^2) = 2F_1(x,Q^2) = \left.\frac{\sigma_T(x,Q^2)}{\sigma_0}\right|_{\gamma^* p} = \sum_i \int_0^1 dz \int_0^1 d\xi\, f_i(\xi)\, \delta(x-\xi z)\left.\frac{\hat\sigma_T(z,Q^2)}{\hat\sigma_0}\right|_{\gamma^* quark}$$

where $\sigma_0 = \frac{4\pi\,\alpha_S^2(Q^2)}{s}$ and similarly for $\hat\sigma_0$ with $\hat s = \xi s$

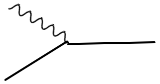
$f_i(\xi)$ is the probability to find in the proton a (« primary ») quark with momentum fraction ξ,

$\hat\sigma_T(z,Q^2)$ is the photon-quark transverse cross section, for a (« secondary ») quark of momentum fraction z,

ξ and z can vary from 0 to 1, but $x = \xi z$ is fixed (hence the δ function)

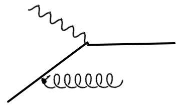
After integration on z : $$2F_1(x,Q^2) = \sum_i \int_0^1 \frac{d\xi}{\xi}\, f_i(\xi)\, \frac{\hat\sigma_T(x/\xi,Q^2)}{\hat\sigma_0}$$

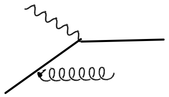
quark evolution equation

At first order : $\gamma^* q \to q$  where $z = x / \xi = 1$

Hence $$2F_1(x,Q^2) = \sum_i \int_0^1 \frac{d\xi}{\xi} f_i(\xi) \frac{\hat{\sigma}_T(x/\xi, Q^2)}{\hat{\sigma}_0}$$

$$\to \sum_i e_i^2 \int_0^1 \frac{d\xi}{\xi} f_i(\xi)\, \delta(1-\frac{x}{\xi}) = \sum_i e_i^2 f_i(x)$$

At next order, the photon quark cross section contains a $\gamma^* q \to q\, g$ contribution 

with for  $$\frac{d\hat{\sigma}}{dp_T^2} \simeq e_q^2\, \hat{\sigma}_0 \frac{1}{p_T^2} \frac{\alpha_s(Q^2)}{2\pi} P_{qq}(z)$$

where $$P_{qq}(z) = \frac{4}{3}\left(\frac{1+z^2}{1-z}\right)$$

$P_{qq}(z)$ is the probability of a quark emitting a gluon and reducing its momentum by the factor z :

« ***splitting function*** »

DGLAP

Thus $$\hat{\sigma}(\gamma^* q \to qg) = \int_{\mu_F^2}^{s^2/4} dp_T^2 \frac{d\hat{\sigma}}{dp_T^2} \simeq e_q^2\, \hat{\sigma}_0\, \frac{\alpha_S(Q^2)}{2\pi} P_{qq}(z)\, \log\frac{Q^2}{\mu_F^2}$$

μ_F = cut off for $p_T \to 0$

and $$\frac{1}{x} F_2(x,Q^2) = \sum_q e_q^2 \int_x^1 \frac{d\xi}{\xi} q(\xi) \left(\delta(1-\frac{x}{\xi}) + \frac{\alpha_s(Q^2)}{2\pi} P_{qq}(\frac{x}{\xi})\, \log\frac{Q^2}{\mu_F^2} \right)$$

logarithmic scaling violation

$$= \sum_q e_q^2 \left[q(x) + \Delta q(x,Q^2) \right]$$

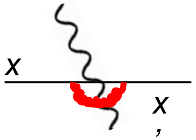
where log dependence is formally absorbed in quark density redefinition

Hence integro-differential ***evolution equation*** for quark distribution :

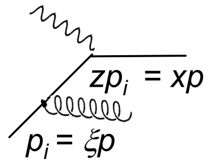
$$\frac{dq(x,Q^2)}{d\log Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \int_x^1 \frac{d\xi}{\xi} q(\xi,Q^2)\, P_{qq}(\frac{x}{\xi})$$

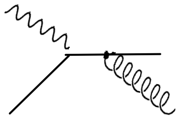
QCD evolution – DGLAP equations (bis)

low Q^2
probes x

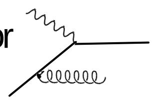


high Q^2
probes
x'<x





$$2F_1(x,Q^2) = \left.\frac{\sigma_T(x,Q^2)}{\sigma_0}\right|_{\gamma^* p} = \sum_i \int_0^1 dz \int_0^1 d\xi\, f_i(\xi)\,\delta(x-\xi z)\left.\frac{\hat\sigma_T(z,Q^2)}{\hat\sigma_0}\right|_{\gamma^* quark}$$

One has for  $\displaystyle \frac{d\hat\sigma}{dp_T^2} \simeq e_q^2\,\hat\sigma_0\,\frac{1}{p_T^2}\,\frac{\alpha_s(Q^2)}{2\pi}\,P_{qq}(z)$ where $\displaystyle P_{qq}(z) = \frac{4}{3}\left(\frac{1+z^2}{1-z}\right)$

$P_{qq}(z)$ is the probability of a quark emitting a gluon of quark momentum factor z : « splitting function »

Hence

$$\hat\sigma(\gamma^* q \to qg) = \int_{\mu_F^2}^{s^2/4} dp_T^2\,\frac{d\hat\sigma}{dp_T^2} \simeq e_q^2\,\hat\sigma_0\,\frac{\alpha_S(Q^2)}{2\pi}\,P_{qq}(z)\,\log\frac{Q^2}{\mu_F^2}$$

μ_F = cut off for $p_T \to 0$ (see below)

$$\frac{1}{x}F_2(x,Q^2) = \sum_q e_q^2 \int_x^1 \frac{d\xi}{\xi}\,q(\xi)\left(\delta\!\left(1-\frac{x}{\xi}\right) + \frac{\alpha_s(Q^2)}{2\pi}\,P_{qq}\!\left(\frac{x}{\xi}\right)\,\log\frac{Q^2}{\mu_F^2}\right)$$

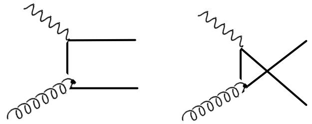
logarithmic scaling violation, formally absorbed in quark density redefinition

DGLAP integro-differential equations

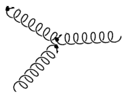
$$\frac{dq(x,Q^2)}{d\log Q^2} = \frac{\alpha_s(Q^2)}{2\pi}\int_x^1 \frac{d\xi}{\xi}\,q(\xi,Q^2)\,P_{qq}\!\left(\frac{x}{\xi}\right)$$

DGLAP equations

Similarly : quark in gluon $P_{q\,g}$



gluon in gluon $P_{q\,g}$



Notation $P_{ij} \otimes f_i(x,Q^2) = \int_x^1 \frac{d\xi}{\xi} P_{ij}(\frac{x}{\xi})\, f_i(\xi,Q^2)$

$$\frac{dq(x,Q^2)}{d\log Q^2} = \frac{\alpha_s(Q^2)}{2\pi}\left[P_{qq} \otimes q(x,Q^2) + P_{qg} \otimes g(x,Q^2)\right]$$

$$\frac{dg(x,Q^2)}{d\log Q^2} = \frac{\alpha_s(Q^2)}{2\pi}\left[P_{gq} \otimes q(x,Q^2) + P_{gg} \otimes g(x,Q^2)\right]$$

Remarks

1. DGLAP equations = Renormalisation group equations (RGE)

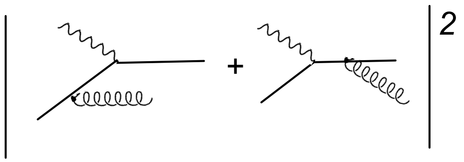
$$q(x,Q^2;\mu_F^2) = q(x) + \frac{\alpha_S(Q^2)}{2\pi}\log\frac{Q^2}{\mu_F^2}\int_x^1 \frac{d\xi}{\xi} P_{qq}\left(\frac{x}{\xi}\right)\; q(\xi)$$

Choice of factorisation scale μ_F is arbitrary → $q(x,Q^2)$ should not depend on μ_F :

$$\frac{dq(x,Q^2;\mu_F^2)}{d\log\mu_F} = 0 \quad \rightarrow \text{ the DGLAP equations}$$

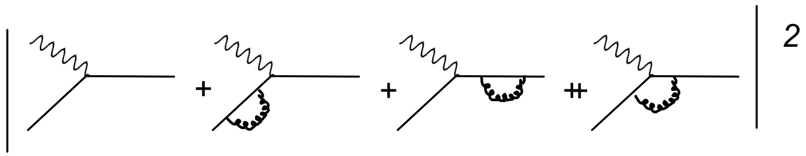
2. Singularities in splitting functions

Remember P_{qq} comes from



and is singular $P_{qq}(z) = \frac{4}{3}\left(\frac{1+z^2}{1-z}\right)$

But interference of virtual corrections with leading order diagram regularise the singularity in P_{qq}



3. Higher orders

NLO and NNLO splitting functions have been calculated. Very complicated !